1.1.

$$\sum_{n=1}^{\infty} \frac{\sin^n x}{3^n} = \sum_{n=1}^{\infty} \left(\frac{\sin x}{3}\right)^n = \lim_{n\to\infty} \frac{\sin x}{3} \frac{1-\left(\frac{\sin x}{3}\right)^n}{1-\frac{\sin x}{3}} = \frac{\sin}{3} \frac{1}{1-\frac{\sin x}{3}} = \frac{\sin x}{3} \frac{3}{3-\sin x} = \frac{\sin x}{3-\sin x}$$

since $\left|\frac{\sin x}{3}\right| < 1$,

d

1.2

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2x+5}{3}\right)^n = \lim_{n\to\infty} \frac{1-\left(\frac{2x+5}{3}\right)^n}{1-\frac{2x+5}{3}}$$

If $\lim_{n\to\infty} \left(\frac{2x+5}{3}\right)^n < \infty$, then $\left|\frac{2x+5}{3}\right| < 1$

$\Rightarrow -3 < 2x+5 < 3$

$-8 < 2x < -2$

$-4 < x < -1$

e

2.1.

Set $f(x) = \frac{(\ln x)^{p-1}}{x}$, then $\sum_{n=2}^{\infty} \frac{(\ln n)^{p-1}}{n} = \int_2^{\infty} \frac{(\ln x)^{p-1}}{x} dx$

set $u = \ln x$, then $\int_{\ln 2}^{\infty} \frac{u^{p-1}}{e^u} de^u = \int_2^{\infty} \frac{u^{p-1}}{e^u} \cdot e^u du$

$\Rightarrow x = e^u$

$$= \int_{\ln 2}^{\infty} u^{p-1} du = \left.\frac{u^p}{p}\right|_{\ln 2}^{\infty}$$

$$= \lim_{u\to\infty} \frac{u^p}{p} - \frac{(\ln 2)^p}{p}$$

$\lim_{u\to\infty} u^p = 0 \Rightarrow p < 0.$

b

3.1.

$f(x) = \frac{1}{(3x+5)^4}$, $\quad R_n \le \int_n^\infty f(x)\,dx = \int_n^\infty \frac{1}{(3x+5)^4}\,dx$

$$= \frac{1}{-9(3x+5)^3}\Big|_n^\infty$$

$$= \frac{1}{9(3n+5)^3} \le \frac{1}{1000}$$

Since when $n=0$, $\frac{1}{9(3\times0+5)^3} = \frac{1}{2025} < \frac{1}{1000}$,

a

then the minimum number of terms is 1.

3.2

$f(x) = \frac{1}{x^4}$, then $R_{10} \le \int_{10}^\infty \frac{1}{x^4}\,dx = \frac{x^{-3}}{-3}\Big|_{10}^\infty = \lim_{x\to\infty} \frac{1}{-3x^3} + \frac{1}{3\times10^3}$

$$= \frac{1}{3000}$$

a

3.3

Alternating Series.

$|R_n| = |S - S_n| \le b_{n+1}$.

$0.001 = \frac{1}{1000}$

Since $\frac{1}{6!} = \frac{1}{720}$, $\frac{1}{7!} = \frac{1}{5040}$,

then $b_6 = \frac{1}{7!} < \frac{1}{1000}$.

Hence, $n = 5$.

d

4.1.

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{2n+1}{4n+3}\right) - \frac{n}{\ln n}$ e

↑ divergent

5.1.

(i) $\frac{x\sin^2 x}{\sqrt[3]{1+x^7}} < \frac{x}{x^{\frac{7}{3}}} = \frac{1}{x^{\frac{4}{3}}}$ convergent

(ii) $\frac{1}{x+e^{2x}} < \frac{1}{x+x^2}$ convergent

(iii) $\frac{x^2}{\sqrt{x^6-1}} > \frac{x^2}{\sqrt{x^6}} = \frac{1}{x}$ divergent

c

6.1

(i) $\frac{n}{n^2+1}\downarrow$, but $\left|(-1)^n\frac{n}{n^2+1}\right|$ divergent (conditional)

(ii) $\frac{n^2}{3^n+2}\downarrow$, and $\left|\frac{n^2}{3^n+2}\right|$ convergent. (absolute)

a.

7.1.

When $x = \frac{1}{3}$, then $\sum_{n=1}^{\infty} (n+1)!\,(3x-1)^n = 0$

When $x \neq \frac{1}{3}$, then $\sum_{n=1}^{\infty} (n+1)!\,(3x-1)^n$ divergent.

the radius is 0.

b

7.2.

$$a_{n+1} = \frac{1\cdot 4\cdot 7 \cdots (3n-2)(3n+1)}{(n+1)!\; 3^{n+1}} \cdot x^{3n+3}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3n+1}{(n+1)\cdot 3}\cdot x^3\right| = |x^3| < 1. \qquad -1 < x < 1$$

radius is 1.

c

7.3.

$$a_{n+1} = \frac{(2x+3)^{n+1}}{2^{n+1}(n+2)}, \qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{2x+3}{2\cdot \frac{n+2}{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2x+3}{2}\right| = \left|x+\frac{3}{2}\right| < 1. \qquad \text{Radius is } 1$$

a.

8.1.

(i) $|a_n| = \left| \frac{n^3}{n^3+2n^2+1} \right| \xrightarrow{n\to\infty} 1.$ divergent

(ii) since $\sin(n\pi) = 0$, then $a_n = 0$.

b,

8.2.

(i) $a_n = \ln \frac{n+1}{2n}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln \frac{n+1}{2n} = \ln \frac{1}{2}$

(ii) $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} \overset{L'H}{=} \frac{\cos x}{1} = 1$

e